Bankruptcy might lead to Pareto Improvement in a Non Convex Economy

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Abstract

We study a finite agent economy with non convex budget set due to the bankruptcy option and characterize condition that guarantee equilibrium existence. It is shown that the choice of the exogenous debt constraint and the seizure rate play an important role in this context. Pareto improvement might be achieved by an authority who regulates the exogenous debt constraint, but letting the debtor choose a too big portfolio might eliminate equilibrium via its effect on the asset price, this threshold depends on the seizure rate that can be interpret as a bankruptcy institution that could define a pro-debtor or pro-creditor law. In bankruptcy, the economy might create a new asset that is better than the exogenous one, leading to a Pareto improvement, furthermore, this bankruptcy equilibrium might be Pareto efficient when the seizure rate is chosen properly.

Keywords: Bankruptcy, general equilibrium, non convexities.

1 Introduction

Consider an individual who goes to a financial institution to ask for a loan, the Bank usually makes a credit analysis of the individual in order to determine the agent’s ability to repay the debt, and fix a credit limit as the maximum amount of money that the Bank should lend to the individual. For simplicity we don’t model the way this limit is estimated and we assume that this limit is exogenously given, that is, it is determined outside the model by some intermediate agent or a regulator.

In certain contexts\textsuperscript{1} we can choose the debt constraint small enough to avoid bankruptcy but big enough such that the constraint is not biding in equilibrium,

\textsuperscript{1}As will be clear latter, in our setting this affirmation is true when we set the seizure rate big enough such that upper constraints is not biding.
so the debt constraint doesn’t bring additional market imperfections and the equilibrium is the same as in a full commitment economy. This case can also be interpreted as a simplification of the Kehoe-Levine model [4] in which debt limit is endogenously determined but an incentive constraint is added in order that the agents’ best response is to choose the portfolio without default.

When the intermediate agent defines the limit of credit, he might not have all the individual’s relevant information, or might have incomplete information about future events and fails to determine a limit of credit below which the debtor always honors her debts. So in certain scenarios the debtor agent might not be able to repay her debt, in this situation we say that she goes bankruptcy.

Allowing for bankruptcy brings some technical difficulties, agent’s feasible set (including consumption and portfolio) is no longer a convex set, so it is possible that the best response correspondence is not a convex set, this prevents us to use the classical fix point theorems to prove equilibrium existence. To overcome this difficulty two seminal papers on the subject [1] and [2] work with a non-atomic set of agents. In a dynamical economy this approach has been used to study some policy implications of a change in the Bankruptcy legislation [3].

When introducing a set of continuous agents, the economy turns out to be convex, in the sense that the aggregate best response correspondence is convex valued, and, except for some technical details, equilibrium existence might be proved in a similar way as in an Arrow-Debreu economy.

In this paper we address an economy with finite agents. As we mentioned, equilibrium might fail to exit due to the non convexities of the feasible set². Through an example we show that when the debtor’s initial wealth is zero we can always find an equilibrium even if the best response correspondence is not convex. The key observation for this is that with the utility specification, debtor’s best response correspondence is inelastic, she chooses between the full commitment portfolio and (if any) the bankruptcy portfolio, in which case she decides to sell as much as she can such that the credit constraint is binding. If credit limit is small enough debtor always choose the full commitment portfolio and if it is large enough she fills for bankruptcy: there would be a threshold of credit limit (independent of asset price) such that debtor goes bankruptcy if, and only if, credit limit is beyond this threshold. To sum up, debtor’s choice, when her initial wealth is zero, depends only on the exogenous debt constraint and we can find equilibrium price from the creditor’s optimization problem.

We also address some welfare properties of the basic example. Regarding the exogenous debt constraint we show that under certain conditions, a regu-

²A source of the non convexities is the fact that agents’ utilities are not defined for negative consumption, in [8] negative consumption is allowed and is interpreted as a penalty, this solves the non convex problem, but hides the role of the seizure rate and the exemptions that are usual in an individual bankruptcy law.
lator who decides the credit limit might Pareto improve both agents. For low values of credit limit the intuition is clear: if we start from a situation in which asset transactions are not allowed (zero credit limit) a slightly increase would allowed both agents to increase consumption and manage risk sharing. A partial equilibrium analysis might get us the wrong idea that increasing the debt constraint always increases the debtor's utility, this is true when prices are fixed since an increase in debt limits enlarges the feasible set, but to complete the picture we need to analyze the general equilibrium effect and look at the price changes when we move the credit limit. If debtor agent fills for bankruptcy she sells as much as she can, an increase in the credit limit allows her to sell more asset, but this has a general equilibrium effect in prices: as debtor agent's default is bigger, a creditor with rational expectation over the default rate is willing to buy the asset only if the interest rate (or the asset price) compensate this increase in default, so in equilibrium prices goes down and the net effect on debtor agent might be negative as we see in a numerical example.

Then, by a slight modification we characterize the cases in which equilibrium exists depending on the value of the exogenous debt constraint. It is shown that for low values of credit limit equilibrium exists with no bankruptcy but if credit limit is too high, a situation that can be interpret as a lack of regularization, equilibrium doesn't exist because debtor agent will chose to short sell as much as she can in the first period and fill for bankruptcy in the second, but this will jeopardize asset price, since agents have rational expectation regarding the rate of default, creditor agent will ask a very law price in order to compensate the bankruptcy, at some point market shut down and equilibrium is not possible.

The paper is organized as follows: next we give a simple example with a creditor and a debtor with null initial wealth, in section 2.4 we show necessary and sufficient conditions for equilibrium existence in the case of two agents and when the debtor's wealth is perfectly correlated with asset returns. In section 4 we address the equilibrium existence problem when debtor's wealth is not perfectly correlated with asset returns such that partial default is allowed, then in section 4.3 through an example we show that a default equilibrium might Pareto dominate a non default equilibrium when the seizure rate is low. In section 5 we show that bankruptcy equilibrium might be Pareto efficient. Finally in section 6 we summarize the main results.

2 A Guiding Example

Consider an economy with 2 agents, named Ms. Moneyless and Mr. Banks. There are two periods \((t = 0, 1)\), one consumption good in each period and one bond that can be negotiated in \(t = 0\) at price \(q\) and promises to pay \(r > 0\) units of consumption in \(t = 1\).

In \(t = 0\) Ms. Moneyless loses all her wealth and starts with \(w_0^M = 0\), so she
is forced to sell the bond in order to have a positive consumption. In \( t = 1 \) her wealth is \( w^M > 0 \) and she is allowed to fill for bankruptcy in case her wealth is not enough to repay the debt, in that case her debt is discharged, but a fraction \( \gamma \in [0,1] \) of her wealth is confiscated to partially repay the debt.

Given the bond price \( q \) and a debt bound \( m > 0 \), Ms. Moneyless optimization problem is:

\[
\text{Maximize } u(qy) + u(\max\{(1-\gamma)w^M; w^M - ry\})
\]  \hspace{1cm} (1)

Note that because of the possibility of bankruptcy we need to introduce an exogenous debt bound, otherwise her best response would be to sell an infinity amount of the bond and fill for bankruptcy.

Mr. Banks has stochastic wealth in \( t = 1 \), his wealth is in the set \( \{w^B_1, \ldots, w^B_S\} \) with probabilities \( \{\pi_1, \ldots, \pi_S\} \) and there is a state \( s \) such that \( w^B_s = 0 \) with \( \pi_s > 0 \), so in \( t = 0 \) he is forced to buy the bond to have a positive consumption in case state \( s \) occurs.

Mr. Banks optimization problem is:

\[
\text{Maximize } u(w^B_0 - qy) + \sum_{s=1}^{S} \pi_s u(w^B_s + r\kappa y)
\]  \hspace{1cm} (2)

where \( \kappa \in [0,1] \) is the mean reimbursement rate, \( \kappa = 1 \) when the debtor honors her debt and \( \kappa < 1 \) otherwise.

An equilibrium for this simple economy is a vector \( (y^M, y^B, q, \kappa) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times [0,1] \), such that:

- Given \( q \), \( y^M \) is an optimal choice for Ms. Moneyless.
- Given \( q \) and \( \pi \), \( y^B \) is an optimal choice for Mr. Banks.
- \( y^M = y^B \) (market clearing)
- \( r\kappa y^M = \min \{ry^M; \gamma w^M\} \) (perfect foresight)

**Remark.** For this example we assume that \( u(x) = \ln(x) \) for both agents. Notice that Ms. Moneyless objective is concave by parts, but not globally concave, so her best response might not be a convex set, so we can’t apply the usual fix points theorems used in general equilibrium theory.
2.1 Borrower’s Problem

Here we characterize Ms. Moneyless optimization problem and we find that her optimal choice depends on the exogenous debt constraint.

Claim 2.1 If $\gamma \geq \frac{1}{2}$, Ms. Moneyless’ optimal utility is given by:

$$v^M(q, m) = \begin{cases} 
\ln (qm(w - rm)), & \text{if } 0 \leq m \leq y^u \\
\ln \left( \frac{q y^u^M}{w} \right), & \text{if } y^u \leq m \leq \bar{m} \\
\ln \left( qm(1 - \gamma)w^B \right), & \text{otherwise}
\end{cases}$$

where $y^u$ is the unconstrained portfolio with full commitment and $\bar{m} > 0$ is independent of $q$.

**Proof** Let’s define $\bar{y} \equiv \frac{2w^M}{r}$ as the discontinuity point in the marginal utility. Note that:

$$u^M(y) = \begin{cases} 
\ln(qy) + \ln(w^M - ry), & \text{if } y \leq \bar{y} \\
\ln(qy) + \ln((1 - \gamma)w^M), & \text{if } y > \bar{y}
\end{cases}$$

If Ms. Moneyless honors her debt, by the first order condition of the unconstrained problem, we obtain that the optimal choice would be $y^u = \frac{2w^M}{r} \leq \bar{y}$, but it may happen that the debt limit doesn’t allow $y^u$, if $m < y^u$, the optimal portfolio should be the extreme point $m$.

Now consider the case $m \geq y^u$, in $[0, \bar{y}]$ the only optimal choice is $y^u$, in $[\bar{y}, m]$ if $m > \bar{y}$ $u^M(\cdot)$ is strictly increasing, so the optimal choice is $m$. To define the optimal portfolio we need to compare $u^M(y^u)$ with $u^M(m)$ when $m > \bar{y}$. Since $u^M(y)$ is strictly increasing in $y \geq \bar{y}$, $u^M(\bar{y}) \leq u^M(y^u)$ and $\lim_{y \to \infty} u^M(y) = +\infty$, by the mean value theorem there is a unique $\bar{m} \geq \bar{y}$ such that $u^M(y^u) = u^M(\bar{m})$ (actually we can compute $\bar{m} = \frac{w^M}{q(1 - \gamma)}$, independent of $q$). Altogether we conclude that agent’s optimal choice $y^M$ is:

$$y^M = \begin{cases} 
m, & \text{if } 0 \leq m < y^u \\
y^u, & \text{if } y^u \leq m < \bar{m} \\
\{y^u, m\}, & \text{if } m = \bar{m} \\
m, & \text{if } m > \bar{m}
\end{cases}$$

In the first two cases she doesn’t fill for bankruptcy, in the third one she is indifferent between filling for bankruptcy or not and in the last case she fills for bankruptcy and sells as much as she can. Computing optimal utilities in each case we get the announced result. 

From the last result we can see that for fixed $m$ Ms. Moneyless bond’s offer is inelastic and with a fixed price $q$ her optimal value $v^M(q, \cdot)$ is a continuous non decreasing function of $m$. This partial result might suggest that Ms. Moneyless is better with a debt bound as large as possible, but in equilibrium her
choices will affect prices, so there is still a general equilibrium effect that we need to consider.

When the debt bound is too low, Ms. Moneyless is actually forbidden to fill for bankruptcy, the threshold is given by \( \hat{m} = \frac{\gamma M}{w} \). It is clear that \( \hat{m} \leq \bar{m} \) with strict inequality if \( \gamma \neq \frac{1}{2} \). So, if \( m \leq \bar{m} \) we already known that the optimal choice is not to fill for bankruptcy, but this could happen for two reasons: If \( m < \hat{m} \) she is just not allowed to do it and if \( \hat{m} \leq m \leq \bar{m} \) although she is allowed to fill for bankruptcy, she chooses to honor her debt because \( m \) is still too low and she can do better with another portfolio.

Claim 2.2 If \( \gamma < \frac{1}{2} \), then Ms. Moneyless chooses to sell as much as possible.

Proof When \( \gamma < \frac{1}{2} \), the unconstrained portfolio \( y = \frac{w^M}{w} \) is not feasible, so if she wants to honor her debt she must choose \( \min \{ m; \hat{m} \} \). If \( m < \hat{m} \), then she is not allowed to fill for bankruptcy and her only possible choice is \( y^M = m \). If \( m \geq \hat{m} \), she chooses \( \hat{m} \) when honoring her debt, but now she is allowed to fill for bankruptcy, so she compares her utility between \( m \) and \( \hat{m} \): \( \max \{ u(q\hat{m}) + u((1 - \gamma)w^M); u(qm) + u((1 - \gamma)w^M) \} \). She clearly chooses \( m \).

Intuitively, when \( \gamma \) is low enough, bankruptcy is a more attractive choice since agents losses are quit small in case of filling for bankruptcy.

2.2 Equilibrium

Once we characterized the debtor’s problem, we find equilibrium for this simple economy.

Claim 2.3 There is an equilibrium for this economy.

Proof First notice that Mr. Banks first order condition is:

\[
-\frac{q}{w^B_0} - qy^B + \sum_{s=1}^{S} \frac{\pi_s K \zeta}{w^T_s + rK\eta^T_s} = 0
\]

Let’s consider three cases:

- Case 1. \( (\gamma \geq \frac{1}{2} \) and \( m < y^b \) or \( (\gamma < \frac{1}{2} \) and \( m \leq \hat{m} \) \)

In this case Ms. moneyless sells as much as she can and she honors her debt because she is not allowed to fill for bankruptcy, so in equilibrium \( y^M = y^b = m, \pi = 1 \) and \( \eta \) is given by:

\[
\eta = \frac{w^B_0 r \zeta}{1 + mr \zeta}
\]

with

\[
\zeta = \sum_{s=1}^{S} \frac{\pi_s}{w^T_s + rm}
\]

Intuitively, when \( \gamma \) is low enough, bankruptcy is a more attractive choice since agents losses are quit small in case of filling for bankruptcy.
Case 2. \( \gamma \geq \frac{1}{2} \) and \( y^u \leq m \leq \overline{m} \).
In this case, the optimal choice for Ms. Moneyless is \( y^M = \frac{w^M}{2r} \) and she honors her debt, so \( \overline{\pi} = 1 \). In equilibrium \( y^B = y^M \), from Mr. Banks first order condition we obtain equilibrium price \( \overline{q} \):
\[
\overline{q} = \frac{2rw^B_W \mathcal{W}}{1 + w^M_W^2}
\]
with
\[
\mathcal{W} = \sum_{s=1}^{S} \frac{\pi_s}{2w^B_s + w^M_s}
\]

Case 3. \( \gamma \geq \frac{1}{2} \) and \( m > \overline{m} \) or \( \gamma < \frac{1}{2} \) and \( m > \hat{m} \). In this case Ms. Moneyless always chooses \( y^M = m \), so in equilibrium \( y^B = m \),
\[
\overline{\pi} = \frac{\gamma w^M}{rm} < 1,
\]
and
\[
\overline{q} = \frac{w^B_r \mathcal{W}'}{1 + mr \mathcal{W}'}
\]
with
\[
\mathcal{W}' = \sum_{s=1}^{S} \frac{\pi_s}{w^B_s + rkm}
\]

Notice that when \( m = \overline{m} \) there are two solutions and agent’s optimal choice is not a convex set, but even in this case we can find an equilibrium.

2.3 Welfare analysis

In this section we concern about the welfare properties of this economy. We suppose that there is an external agent who regulates the debt limits, a natural question is whether this regulator can improve agents utilities by increasing the debt limit.

Claim 2.4 A policy that regulates the debt limit \( m \) is effective only when there is no bankruptcy and debtor sells as much as she can.

Proof We will show that in equilibrium a policy that regulates \( m \) is effective only in case 1 of claim 2.3. It is clear that in case 2 a policy is neutral because all equilibrium variables are independent of \( m \), so any change that keeps \( m \) in the relevant interval of case 2 doesn’t change optimal utilities.

In case 3 Ms. Moneyless chooses to sell as much as she can and she fills for bankruptcy, in equilibrium her utility depends only in the product \( qm \). Mr. Banks optimal utility also depends on \( qm \) and on \( \overline{\pi} mr \). From claim (2.3) we can
see that in equilibrium $q_m$ depends on $\kappa m r$ and that this last term is equal to $\gamma w^M$, which is fixed, so any change in $m$ won’t change agents optimal utilities in equilibrium.

Now consider case 1. Ms. Moneyless is not allowed to fill for bankruptcy, but as was seen in claim (2.2), she still chooses $m$, but she fully pays back in period $t = 1$, so $\overline{\pi} = 1$. Ms. Moneyless and Mr. Banks optimal values in equilibrium are, respectively:

\[ v^M(m) = \ln(\overline{\pi} m) + \ln(w^M - rm), \]
\[ v^B(m) = \ln(w^B_0 - \overline{\pi} m) + \sum_{s=1}^{S} \pi_s \ln(w^B_s + rm) \]

with $\overline{\pi}$ as in claim (2.3).

\[ \frac{dv^B}{dm} = -\frac{1}{w^B_0 - \overline{\pi} m} \frac{d(\overline{\pi} m)}{dm} + \sum_{s=1}^{S} \frac{\pi_s r}{w^B_s + rm} \]

then, using Mr. Bank’s first order condition:

\[ \frac{dv^B}{dm} = -\frac{1}{w^B_0 - \overline{\pi} m} dq(m) \]

but with $\overline{\pi}(m)$ given as in claim (2.2) we obtain that $\overline{\pi}(m)$ is decreasing in $m$, so the last equality yields a positive relationship between Mr. Banks optimal utility in equilibrium and the bounded short sale $m$ of Ms. Moneyless. In the other hand Ms. Moneyless optimal utility in equilibrium is not monotone with $m$:

\[ \frac{dv^M}{dm} = \frac{1}{\overline{\pi}} \frac{d(\overline{\pi} m)}{dm} - \frac{r}{w^M - rm} \]

and we notice that:

\[ \text{Sign} \left( \frac{d(\overline{\pi} m)}{dm} \right) = \text{Sign} \left( W' + m \frac{dW'}{dm} \right) \]

after some algebra we can see that $\frac{d(\overline{\pi} m)}{dm} > 0$. So, an increase in $m$ increases Ms. Moneyless utility in $t = 0$, but decreases her utility in $t = 1$.

In the last claim we showed that when debtor borrows as much as possible, but doesn’t fill for bankruptcy, a policy maker who decides the bounded short sale $m$ can affect agent’s utilities by making infinitesimal changes. When $m$ is small enough increases in $m$ will lead to Pareto improvement, but after some value of $m$ debtor’s utility will decrease while creditor’s utility still increases as can be seen in figure 1.

It can be seen that for low values of $m$ both agents improve their utilities when the debt limit $m$ is increased, but after some value Ms. Moneyless utility
decreases, while Mr. Banks utility increases for all values of $m$. As mentioned before, in a partial equilibrium analysis (fixing $q$) debtor’s optimal utility is non decreasing in $m$, but in a general equilibrium scheme, increasing $m$ has negative effects on prices that tends to diminish optimal utility and for some values it can more than compensate the utility gains of increasing $m$ such that the net effect leads to utility loses for the borrower.

2.4 Equilibrium Existence and Non-Existence when $w^M_0 > 0$

In the last section we presented a bankruptcy model with two agents and showed that for that simple example there is always an equilibrium and the bankruptcy decision depends on the debt limit $m$: if it is less than $\overline{m}$ debtor always honors her debt, but if it is grater than $\overline{m}$ then she prefers to fill for bankruptcy.

In this section we will give a counterexample of equilibrium existence and as we shall see later, a crucial assumption in the simple example is that debtor’s initial wealth is null in $t = 0$, with the utility function specified it is straightforward that bond offer is inelastic and that $\overline{m}$ is independent of price $q$. This assumption allows us to examine if the exogenous debt limit $m$ is grater than $\overline{m}$ before solving agent’s optimization problem.

Consider the model in the last section, with the only difference that Ms. Moneyless wealth in $t = 0$ is $w^M_0 > 0$. So, given price $q$ and the debt bound $m$ her optimization problem is:

$$\text{Maximize } u(w^M_0 + qy) + u(\max \{(1 - \gamma)w^M; w^M - ry\}) \quad (6)$$
If we still consider log utilities, then the unconstrained portfolio would be:

\[ y_u(q) = \frac{qw^M - rw_0^M}{2qr} \]  

(7)

if prices are too low, then she prefers to save instead of borrowing, and it is also clear that if initial wealth is null then we recover the inelastic bond offer.

Mr. Banks’ optimization problem is the same as in the last section, he is the natural borrower in this economy and from the last observation we see that if Ms. Moneyless is not allowed to fill for bankruptcy or if the debt limit is below the indifference threshold then \( w_0^M \) must be low enough to guarantee that \( y_u \) is positive.

For simplicity we will assume that \( \gamma \geq \frac{1}{2} \), so \( y_u \leq \bar{m} \). Using creditor’s first order condition we find that equilibrium price \( q_u \) in case of no bankruptcy must solve:

\[ \frac{1}{2rw_0^B - qw^M + rw_0^M} = \sum_{s=1}^{S} \frac{\pi_s}{2qw^B_s + qw^M - rw_0^M} \]  

(8)

As in the last section we need to find the indifference debt limit \( \overline{m} \) as the bound that left the debtor indifferent between filling for bankruptcy and fully repaying.

For a given price \( q \), \( \overline{m} \) must satisfy:

\[ \ln (w_0^M + qy(q)) + \ln (w^M - ry_u(q)) = \ln (w_0^M + qm) + \ln ((1-\gamma)w^M) \]

After some algebra we get:

\[ \overline{m}(q) = \left( \frac{\gamma}{1-\gamma} \right) \left[ \frac{(y_u(q))^2}{\bar{m}} + \frac{w_0^M}{q} \right] \]  

(9)

Note that if \( w_0^M = 0 \) then \( \overline{m} \) is independent of \( q \).

Given the exogenous debt limit \( m \), if \( m \leq \bar{m}(q_u) \) then debtor agent optimal choice is the unconstrained portfolio \( y_u(q_u) \) and even if \( \bar{m} \leq m \leq \overline{m}(q) \) (bankruptcy is allowed) she is better with that portfolio, so \( (y_u, q_u, \kappa = 1) \) is the equilibrium for this economy.

Numerically we can solve (8) by Newton’s Method, then compute \( y_u(q_u) \) by (7) and with finally \( \overline{m} \) with (9) and compare it with \( m \).

What if \( m > \overline{m}(q_u) \)? In this case, given the price \( q_u \), debtor agent prefers to fill for bankruptcy and chooses the portfolio \( m \), so there is no equilibrium with full commitment and we need to search for an equilibrium with bankruptcy.
In a bankruptcy equilibrium \( \kappa = \frac{\hat{m}}{m} < 1 \), debtor chooses \( m \) and by creditor’s first order condition price in bankruptcy \( q^B \) is:

\[
q^B = \frac{\sum_{s=1}^{S} \pi_s w^B_s}{1 + m \sum_{s=1}^{S} \pi_s w^B_s + r \kappa m}
\]

Then we need to compute the new value of \( m \) with price \( q^B \), and compare it with \( m \). If \( m > \overline{m}(q^B) \) then there is an equilibrium with bankruptcy, but if \( m < \overline{m}(q^B) \) then there is any equilibrium.

We will write \( \overline{m}(\cdot) \) as a function of the exogenous debt constraint \( m \) as follows: given a portfolio \( y = m \) offered by the debtor, the borrower is willing to accept this portfolio only if price \( q(m) \) solves:

\[
-\frac{q}{w^B_0 - qm} + \sum_{s=1}^{S} \frac{\pi_s K_r}{w^B_s + r \kappa m} = 0
\]

with \( rm \kappa = \min \{ rm; \gamma w^M \} \).

If price is \( q(m) \), then the unconstrained optimal portfolio for debtor agent \( y(q(m)) = y(m) \) is given by (7) with \( q = q(m) \). With \( y(m) \) and \( q(m) \) we compute \( \overline{m}(m) \) as in (9).

Define the correspondence \( M : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) as \( M(m) = [\overline{m}(q^u), \overline{m}(m)] \)

**Theorem 2.5** There is an equilibrium for this economy if, and only if, \( m \) is not an interior point of \( M \).

**Proof**:

\( \Rightarrow \) If \( m \in \hat{M}(m) \) then \( m > \overline{m}(q^u) \), then when price is \( q^u \) debtor prefers to fill for bankruptcy and chooses portfolio \( m \), so there is not an equilibrium with full commitment.

In the other hand, if debtor chooses portfolio \( m \) and fills for bankruptcy, creditor is willing to buy this portfolio at price \( q^m \), but with this price the threshold of indifference debt limit is \( \overline{m}(m) \), since \( m < \overline{m}(m) \) then debtor agent prefers not to fill for bankruptcy and chooses the unconstrained portfolio \( y(m) \), but creditor is not willing to buy this portfolio at price \( q(m) \), so there is not an equilibrium with bankruptcy.

\( \Leftarrow \) If \( m \leq \overline{m}(q^u) \) then for price \( q^u \) debtor optimal choice is \( y^u \) and this portfolio is supported by creditor, so this is an equilibrium with full commitment.

If \( m \geq \overline{m}(m) \) then the only possible equilibrium is one with bankruptcy, in this case debtor chooses portfolio \( m \) and fills for bankruptcy, creditor buys this
portfolio only if price is \( q(m) \) and with this price the threshold of indifference is \( \overline{m}(m) \leq m \) so debtor agent has no incentives to change his portfolio. \( \blacksquare \)

In figure 2 it is shown the graph of \( M \) for four different values of the initial wealth \( w_0^M \). As we mentioned before, if \( w_0^M = 0 \) then the threshold of indifference is independent of price and so is of \( m \), then \( M \) is actually a single line with empty interior so there is equilibrium for any value of the exogenous debt limit. Note that when \( w_0^M > 0 \) the correspondence has no empty interior. For a fixed \( w_0^M \) its observed that there is an interval of values of \( m \) for which there is equilibrium with full commitment, then we observe a transition interval for which there is no equilibrium, after this transition we find an interval in which there is equilibrium with bankruptcy and for large values of \( m \) there is no equilibrium.

The size of this intervals depends on the initial wealth \( w_0^M \). For instance, if it is small enough, the transition interval is very small (in the figure is it almost indistinguishable) and the interval for bankruptcy equilibrium is quit big.

The interval of bankruptcy equilibrium also seems to decrease with \( w_0^M \), as a matter of fact there is a point \( (w_0^M \sim 0.1712) \) such that this interval is reduced to a single point and for values above this, the interval is empty, so there is no equilibrium with bankruptcy (see d) in the figure).

Before we formalize this observation we need the following technical result regarding some properties of \( \overline{m}(m) \).

**Lemma 2.6** Define \( f(m) = \overline{m}(m) - m \), then

\(\text{(a)}\) \( \overline{m}(\cdot) \) is a continuous, strictly increasing and strictly convex function for \( m > \hat{m} \).

\(\text{(b)}\) There is some \( m_0 > 0 \) such that \( f(m) > 0 \) for all \( m \geq m_0 \).

\(\text{(c)}\) If there is a \( m \) with \( f(m) < 0 \) then, there are two roots \( a < m < b \) for \( f \).

\(\text{(d)}\) \( f \) has no more than two roots.

**Proof** :

\(\text{(a)}\) Note that \( \overline{m}(\cdot) \) is differentiable for all \( m \neq \hat{m} \), so

\[
\frac{d \overline{m}(m)}{dm} = \frac{\gamma}{1 - \gamma} \left[ \frac{2y(m)}{\hat{m}} \frac{dy(q(m))}{dq} - \frac{w_0^B M(q(m))}{q^2(m)} \right] \frac{dq(m)}{dm},
\]

\[
\frac{dy(q)}{dq} = \frac{w_0^B}{2q^2},
\]
then

\[
\frac{d \overline{m}(m)}{dm} = \frac{\gamma}{1 - \gamma} \frac{w_0^B}{2q^2} \left[ \frac{y(m)}{\hat{m}} - \left( \frac{m}{\hat{m}} - 1 \right) \right] \frac{dq(m)}{dm}. \tag{11}
\]
$q(m)$ solves:

$$F(q, m) = \left( 1 + mrK \sum_{s=1}^{S} \frac{\pi_s}{w_s^B + rkm} \right) q - w_0^B \sum_{s=1}^{S} \frac{\pi_s}{w_s^B + rkm} = 0.$$ 

So, by the implicit function theorem:

$$\frac{dq(m)}{dm} = -\frac{F_m(q(m), m)}{F_q(q(m), m)}$$

with

$$F_q(q(m), m) = \left( 1 + mrK \sum_{s=1}^{S} \frac{\pi_s}{w_s^B + rkm} \right) > 0$$

so $\text{Sign} \left( \frac{dq(m)}{dm} \right) = -\text{Sign} \left( F_m(q(m), m) \right)$.

$$F_m(q(m), m) = \begin{cases} 
    rq(m) \sum_{s=1}^{S} \frac{\pi_s}{w_s^B + rkm} + (w_0^B - mq(m))r^2 \sum_{s=1}^{S} \frac{\pi_s}{(w_s^B + rkm)^2} & \text{if } m < \hat{m} \\
    \frac{w_0^B}{m^2} \sum_{s=1}^{S} \frac{\pi_s}{w_s^B + rkm} - \frac{r^2}{m^2} \sum_{s=1}^{S} \frac{\pi_s}{(w_s^B + rkm)^2} & \text{if } m > \hat{m}
\end{cases}$$

Note that $mq(m) < w_0^B$, so $F_m(q(m), m) > 0 \Rightarrow \frac{dq(m)}{dm} < 0$ and since $y(m) < \hat{m}$ we show that $\overline{m}(\cdot)$ is strictly increasing.

To see that $\overline{m}(\cdot)$ is strictly convex in $m > \hat{m}$, it suffices to prove that $\frac{d^2\overline{m}(m)}{dm^2} > 0$.

For all $m > \hat{m}$:

$$\frac{1}{q^2(m)} \frac{dq(m)}{dm} = \text{cte} < 0$$

So, by (11)

$$\text{Sign} \left( \frac{d^2\overline{m}(m)}{dm^2} \right) = -\text{Sign} \left( \frac{dq(m)}{dm} \right) = -\text{Sign} \left( \frac{dy(m)}{dm} \right) = -\text{Sign} \left( \frac{dq(m)}{dm} \right).$$

(b) First we see that

$$\lim_{m \to +\infty} y(m) = \lim_{m \to +\infty} y(q(m)) = y \left( \lim_{m \to +\infty} (q(m)) \right),$$
Figure 2: The graph of the correspondence $M$ for different values of the debtor’s initial wealth in $t = 0$: a) $w_0^M = 0.01$, b) $w_0^M = 0.15$, c) $w_0^M = 0.1712$ and d) $w_0^M = 0.2$
where the last equality comes from continuity of $y(\cdot)$. From (10) it is straightforward that \( \lim_{m \to +\infty} q(m) = 0 \) and by (7) \( \lim_{q \to 0} y(q) = -\infty \). Using (11) we conclude that \( \lim_{m \to +\infty} \overline{m}(m) = +\infty \).

So, there is an \( m_0 > 0 \) such that \( f'(m_0) > 0 \).

By convexity, for all \( m \):

\[
 f(m) \geq f(m_0) + f'(m_0)(m - m_0) \to +\infty \text{ as } m \to +\infty.
\]

Increasing \( m_0 \) (if necessary) we obtain the claimed result.

(c) First note that \( \overline{m}(y^u) = \overline{m}(q^u) \), and using item a) we have that for all \( m \in (y^u, \overline{m}(q^u)] \), \( \overline{m}(m) > \overline{m}(q^u) > m \), so \( f(m) > 0 \). That is, all \( m \) such that \( f(m) \leq 0 \) (including the roots of \( f \)) must be greater than \( \overline{m}(q^u) \).

Now take an \( m \) such that \( f(m) < 0 \), since \( m > \overline{m}(q^u) \) and \( f(m(q^u)) > 0 \), by continuity of \( f \) there is some \( \overline{m}(q^u) < a < m \) such that \( f(a) = 0 \).

By item b) there is some \( m_0 > m \) such that \( f(m_0) > 0 \), then by continuity of \( f \) there must be an \( m < b < m_0 \) such that \( f(b) = 0 \).

(d) Let \( a \) and \( b \) be two roots of \( f \) and suppose there is another root \( c \), so, one of them might be written as a convex combination of the other two, WLG suppose \( c = \alpha a + (1 - \alpha)b \) for some \( \alpha \in (0, 1) \), since \( f \) is strictly convex:

\[
 0 = f(c) = f(\alpha a + (1 - \alpha)b) < \alpha f(a) + (1 - \alpha)f(b) = 0.
\]

This contradiction shows that \( f \) has no more than two roots.

**Proposition 2.7** For the economy with exogenous debt limit \( m \):

(a) Given \( w_0^M > 0 \) there are intervals \( I^{NB} \), \( I^T \), \( I^B \) and \( I^{NE} \) such that the space of exogenous debt limits \( \mathbb{R}^+ = I^{NB} \cup I^T \cup I^B \cup I^{NE} \). In \( I^{NB} \) there is equilibrium with no bankruptcy, in \( I^B \) there is equilibrium with bankruptcy and in \( I^T \cup I^{NE} \) there is no equilibrium with \( I^T \subset \text{conv}(I^{NB} \cup I^B) \), that is, \( I^T \) is a transition interval between equilibrium with full commitment and equilibrium with bankruptcy.

(b) The interval of bankruptcy equilibrium \( I^B \) is non increasing with \( w_0^M \).

(c) There is a \( \overline{w}_0^M > 0 \) such that \( I^B = \emptyset \) for all \( w_0^B > \overline{w}_0^M \) and \( I^B \neq \emptyset \) for all \( w_0^M \leq \overline{w}_0^M \).

**Proof :**
(a) It is clear that $m \notin \hat{M}(m)$ for all $m \leq \overline{m}(q^a)$, so there is an equilibrium with no bankruptcy for all such $m$. Define $I^{NB} = (0,\overline{m}(q^a)]$.

Now we consider 3 cases:

(i) $f$ has no roots.
By lemma 2.6 $f(m) > 0$ for all $m$. Take any $m > \overline{m}(q^a)$, then $f(m) = \overline{m}(m) - m > 0$, so $\overline{m}(q^a) < m < \overline{m}(m)$, that is $m \in M(m)$ and by theorem 2.5 we conclude that there is no equilibrium for such $m$. Define $I^T = I^B = \emptyset$ and $I^{NE} = (\overline{m}(q^a), +\infty)$.

(ii) $f$ has only one root.
Let $a$ be the only root of $f$, then by the lemma we have that $f(m) \geq 0$ for all $m$ with equality only at $a$ and by the proof of the lemma we known that $a > \overline{m}(q^a)$. Define $I^T = (\overline{m}(q^a), a)$, $I^B = \{a\}$ and $I^{NE} = (a, +\infty)$. It is clear that for all $m \in I^T \cup I^{NE}$ there is no equilibrium because $m$ is an interior fix point of $M$ for such $m$. For $m = a$ there is an equilibrium with bankruptcy.

(iii) $f$ has two roots.
Let $a < b$ be the two roots of $f$, define:

$$I^T = (\overline{m}(q^a), a),$$

$$I^B = [a, b]$$

and

$$I^{NE} = (b, +\infty).$$

Take any $m \in I^T \cup I^{NE}$ then $f(m) > 0$: If $f(m) < 0$ by lemma 2.6 there would be two roots of $f$ with $a' < m < b'$ which is a contradiction with the fact that $f$ has no more than two roots, by the same reason $f(m) \neq 0$. Then $m$ is an interior fix point of $M$ and there is no equilibrium for such $m$.

Now take any $m \in I^B$ then there is some $\alpha \in [0, 1]$ such that $m = \alpha a + (1 - \alpha) b$, by the strict convexity of $f$:

$$f(m) = f(\alpha a + (1 - \alpha) b) \leq \alpha f(a) + (1 - \alpha) f(b) = 0$$

with strict inequality for $\alpha \in (0, 1)$. So $m \geq \overline{m}(m)$ for such $m$ which implies that $m$ is not an interior fix point of $M$ and by theorem 2.5 there is an equilibrium with bankruptcy.

(b) We write $f$ explicitly as a function of $m$ and $w_0^M$ as $f(m, w_0^M) = \overline{m}(m, w_0^M) - m$. Note that $q(m)$ doesn’t depend on $w_0^M$, so $\overline{m}(\cdot)$ depends on $w_0^M$ directly by (9) and indirectly through $y(\cdot)$:

$$\frac{\partial f(m, w_0^M)}{\partial w_0^M} = \frac{\gamma}{1 - \gamma} \left[ \frac{2y(m, w_0^M)}{m} \left( \frac{\partial y(m, w_0^M)}{\partial w_0^M} \right) \right] + \frac{1}{q(m)}.$$
then, using (7)
\[
\frac{\partial f(m, w_0^M)}{\partial w_0^M} = \frac{\gamma}{1 - \gamma} \left[ 1 - \frac{y(m, w_0^M)}{m} \right] \frac{1}{q(m)} > 0.
\]

So, for a fix \(m\), \(f\) increases with \(w_0^M\). Consider the same tree cases as in the previous item:

(i) \textbf{\textit{f has no roots.}}

In this case \(I^B = \emptyset\) and \(f(m, w_0^M) > 0\) for all \(m\), so an increase in \(w_0^M\) increases \(f\) for all \(m\) and left \(I^B\) empty.

(ii) \textbf{\textit{f has only one root.}}

Let \(a\) be a root of \(f\), then \(f(m, w_0^M) \geq 0\) for all \(m\) with equality only in \(a\). In this case \(I^B = \{a\}\), an increase in \(w_0^M\) makes \(f\) strictly positive for all \(m\) and reduces \(I^B\) to the empty set.

(iii) \textbf{\textit{f has two roots.}}

Let \(a < b\) be two root of \(f\), in this case \(I^B = [a, b]\). Take any \(m \in (a, b)\).

By item \(a)\) we have that \(f(m) < 0\). By convexity:

\[
0 > f(m) \geq f(a) + f'(a)(m - a) = f'(a)(m - a)
\]

\[
0 > f(m) \geq f(b) + f'(b)(m - b) = f'(b)(m - b)
\]

So \(f'(a) < 0\) and \(f'(b) > 0\). By the implicit function theorem:

\[
\frac{da}{dw_0^M} = -\frac{\frac{\partial f(a, w_0^M)}{\partial w_0^M}}{\frac{\partial f(a, w_0^M)}{\partial m}} > 0,
\]

and

\[
\frac{db}{dw_0^M} = -\frac{\frac{\partial f(b, w_0^M)}{\partial w_0^M}}{\frac{\partial f(b, w_0^M)}{\partial m}} < 0
\]

So, an increase in \(w_0^M\) increases \(a\) and reduces \(b\), then the interval \(I^B\) reduces.

(c) Consider the following problem:

\[
\begin{align*}
\text{maximize} & \quad w \\
\text{subject to} & \quad f(m, w) \leq 0 \\
& \quad m(y^u, w) - m \leq 0
\end{align*}
\]

We already known that \(f\) is strictly convex in \(m\); using (9) and (7) it is easy to verify that \(f\) is also strictly convex in \(w\). The feasible set is closed by continuity; suppose it is not bounded, then there is some sequence \((m_t, w_t)\) in the feasible set such that \(||(m_t, w_t)|| \rightarrow \infty\). By item \(b)\) of lemma 2.6, for
a fixed \(\bar{w}\) there is a \(\bar{m}\) such that \(f'_m(\bar{m}, \bar{w}) > 0\).

By convexity and feasibility we have that:

\[
0 \geq f(m_t, \bar{w}) \geq f(\bar{m}, \bar{w}) + f'_m(\bar{m}, \bar{w})(m_t - \bar{m}) + f'_w(\bar{m}, \bar{w})(w_t - \bar{w}),
\]

by the last item we known that \(f'_w > 0\) so we reach a contradiction for \(t\) large enough.

Since the objective function is continuous and the feasibility set is compact there is an optimal vector \((m^*, w^*)\).

Fix any \(w\), then if \(m = \overline{m}(y^w, w) > y^w\), by lemma 2.6 we have that \(\overline{m}(\cdot, w)\) is strictly increasing, so \(\overline{m}(m, w) > \overline{m}(y^w, w) = m \Rightarrow f(m, w) > 0\), that is, \((m, w)\) is not feasible, which implies that the second constraint of (12) is not binding.

Taking \(w = 0\) we have that \(f(m, 0) < 0\) for all \(m > \overline{m}(y^w, 0)\), so, by continuity we can increase \(w\) keeping feasibility, then the non negative constraint in \(w\) is not binding.

By the above consideration we can restrict our self to the following problem:

\[
\begin{align*}
\text{maximize} & \quad w > 0, m \\
\text{subject to} & \quad f(m, w) \leq 0
\end{align*}
\]  

Taking a \(w_0 > 0\) small enough, there is an \(m_0\) such that \(f(m_0, w_0) < 0\) and since \(f\) is convex, the Slater qualification constraints are satisfied, moreover since the objective function is linear, \((m^*, w^*)\) is an optimal vector if, and only if, there is some \(\mu \geq 0\) such that:

\[
\begin{align*}
1 - \mu f'_w(m^*, w^*) &= 0, \\
\mu f'_m(m^*, w^*) &= 0, \\
f(m^*, w^*) &\leq 0 \text{ and} \\
\mu f(m^*, w^*) &= 0.
\end{align*}
\]

It is clear that \(\mu\) can’t be zero, so

\[
\begin{align*}
f(m^*, w^*) &= 0, \\
f'_m(m^*, w^*) &= 0 \text{ and} \\
\mu &= \frac{1}{f'_w(m^*, w^*)} > 0.
\end{align*}
\]
Suppose \((m', w')\) is another optimal vector, then it is clear that \(w' = w^*\); \(m'\) must satisfy
\[
f'_m(m', w^*) = f'_m(m^*, w^*) = 0
\]
Since \(f\) is strictly convex we conclude that \(m' = m^*\), so there is a unique solution.

Finally we prove that \(\pi_0^M = w^*\) satisfies our claim:

\(m^*\) is a root of \(f\) and by the previous item it is the only root, otherwise \(f'_m(m^*, w^*) \neq 0\). \(f(m, w^*) \geq 0\) for all \(m\) with equality only at \(m^*\), \(I^B = \{m^*\}\) is the set where equilibrium with bankruptcy exists. Take any \(w^M_0 > w^*\), since \(f(m, \cdot)\) is strictly increasing we conclude that \(f(m, w^M_0) > 0\) for all \(m\), so the set of equilibrium with bankruptcy is reduced to the empty set.

Now take any \(w^M_0 < w^0\) then \(f(m^*, w^M_0) < 0\) and by lemma 2.6 there would be two roots of \(f\): \(a\) and \(b\) with \(a < m^* < b\) and \(I^B = [a, b] \neq \emptyset\).

### 2.4.1 Eliminating the transition interval

We just showed that when the seizure rate is \(\gamma \geq \frac{1}{2}\), then for debt constraints \(m > \bar{m}(q^u)\), there might be what we called a transition interval, such that is not possible to pass smoothly from a non-default equilibrium to a default equilibrium. If we take a lower seizure rate \((\gamma < \frac{1}{2})\) it is possible to eliminate this awkward interval as follows:

Note that given an asset price \(q\), debtor agent (Ms. Moneyless) chooses the portfolio:
\[
y(q) = \min \left\{ \frac{\bar{m}}{2\gamma} - \frac{w^0_0}{2q}; \bar{m} \right\}
\]
When \(\gamma \geq \frac{1}{2}\) the minimum is attained in the interior; but if we consider \(\gamma < \frac{1}{2}\) the solution would be at the interior provided the price is low enough. In equilibrium it depends on the endowments distribution: if the debtor agent is poor enough in the first period, then the constraint \(y \leq \bar{m}\) would be binding, specifically, if \(w^M_0\) satisfies:
\[
\frac{W}{1 + \bar{m}W} w^M_0 > \frac{\gamma}{1 - 2\gamma} \frac{w^M_0}{\bar{m}}
\]
then the non-default equilibrium portfolio is \(y^u = \bar{m}\), so \(\bar{m}(q^u) = \bar{m}\).
Figure 3: *Graph of $M$ for $\gamma = 0.25$ and $w_0^M = 0.15$. There is no transition interval*.

For $m > \hat{m}$, but close enough to $\hat{m}$, the asset price in the default equilibrium $q(m)$ would be such that agent optimal portfolio in $y \leq \hat{m}$ is still $\hat{m}$, so $\hat{m}(m) = \hat{m}$ for all such $m$. For large $m$, asset price $q(m)$ would be low enough such that the optimal portfolio in $y \leq \hat{m}$ would be interior and the function $m(\cdot)$ (defined in the previous section) would have the behavior described in lemma 2.6. In figure 3 the transition interval vanishes, the function $\hat{m}(\cdot)$ is constant for values close enough to $\hat{m}$ and then it has the same behavior showed in the previous section. In this case it is possible to pass smoothly from a non-default equilibrium to a default equilibrium, but as we notice, for large values of $m$ equilibrium existence fails.

### 2.4.2 Default equilibrium for high debt constraints

To this point we’ve showed, for the guiding example, that when the debt constraint $m$ is too high, then equilibrium existence fails, independently of the seizure rate. Actually we can show that defining exogenously the debtor identity, equilibrium might exists for high values of $m$.

Choosing appropriately the endowment distribution (as we did in the numerical examples) we can identify a natural lender and borrower when there is not default. In this situation, a non-negativity constraint $y \geq 0$ for agent $M$ (Ms. Moneyless) is not biding, thus irrelevant in the non-default equilibrium, but the picture changes as we allow filling for bankruptcy and a high debt constraint. Notice that when $m$ is high enough, the default price $q(m)$ is so low that our natural borrower prefers a negative portfolio in $y \leq \hat{m}$, that is, she prefers to save and becomes a lender.

To avoid that the natural borrower eventually prefers to become a lender we should introduce the non-negativity constraint when we solve her optimization problem in $y \leq \hat{m}$. Given an asset price $q$, portfolio $y = 0$ is optimal in $[0, \hat{m}]$ if, and only if,

$$ q \leq r \frac{w_0^M}{w^M}. $$
Figure 4: *The graph of the correspondence $M$ when the natural borrower is restricted to sell the asset for all prices.* a) $w_0^M = 0.15$, $\gamma = 0.70$ and b) $w_0^M = 0.25$, $\gamma = 0.40$

So, for a large debt constraint $m$, debtor agent chooses the null portfolio. Then, $\bar{m}(m)$ is given by:

$$\bar{m}(m) = \frac{\gamma w_0^M}{1 - \gamma q(m)},$$

where $q(m)$ is the asset price that the lender is willing to accept if portfolio is $y = m$, and is defined as:

$$q(m) = \frac{C}{m}$$

with $C = \gamma w^M w_0^R \sum_s \left( \frac{\pi_s}{\sqrt{\gamma w}} \right)$

then, $\bar{m}(\cdot)$ is a linear function for $m$ large enough:

$$\bar{m}(m) = \frac{\gamma w_0^M}{1 - \gamma C} m.$$

So, if $w_0^M$ is small enough (debtor is poor enough in $t = 0$), $\bar{m}(\cdot)$ has a slope less than 1 and it never intersect the 45° line³ as is shown in figure 4. As it can be seen there is default equilibrium for large $m$, if $\gamma < \frac{1}{2}$ then we can eliminate the transition interval and remarkably there is equilibrium for any debt constraint!!.

### 3 A Bankruptcy Model with Two Agents

Consider an economy with two agents $i = 1, 2$, two periods $t = 1, 2$ and $S$ states of nature in $t = 1$. There is only one consumption good in this economy and

³Note that we must consider the debt constraint $m$ from which $\bar{m}(\cdot)$ becomes linear. It should be before the concave section of $\bar{m}(\cdot)$ hits the 45° line; this is guaranteed if $w_0^M$ is small enough.
agents can use an asset with nominal payoffs \( r \in \mathbb{R}^S_+ \setminus \{0\} \) to smooth consumption.

To fix ideas we consider that only agent 1 is allowed to fill for bankruptcy in case she can’t pay her debt\(^4\). When she fills for bankruptcy, the bankruptcy procedure seize a portion \( \gamma \in [0, 1] \) of her wealth. Then she fills for bankruptcy when the value of the debt is greater than the confiscable wealth.

We assume that agents’ preferences are described by expected utilities with utilities indexes \( u^i : \mathbb{R}_+ \to \mathbb{R}_+ \) in\(^5\) \( C^3 \), strictly concave, strictly increasing with \( u^i(x) \to +\infty \) when \( x \to +\infty \) and satisfy Inada conditions\(^6\):

\[
\lim_{x \to 0} \frac{du^i(x)}{dx} = +\infty
\]

and

\[
\lim_{x \to +\infty} \frac{du^i(x)}{dx} = 0
\]

Agent 1 solves the following optimization problem:

\[
\max_{y \leq \hat{m}} u^1(w^1_0 + qy) + \sum_{s=1}^S \pi_s u^1(\max\{w^1_s - r^s y; (1 - \gamma)w^1_s\}) \tag{14}
\]

For simplicity in this section we assume that \( w^1_s r^s \) is constant, so there is no partial default, if agent 1 defaults then she defaults in all states of nature.

Agent 2 correctly anticipates agent 1’s default and she takes the mean reimbursement rate \( \kappa \in [0, 1] \) as given and solves the following optimization problem:

\[
\max_{y \in \mathbb{R}} \ u^2(w^2_0 - qy) + \sum_{s=1}^S \pi_s u^2(w^2 + \kappa r^s y) \tag{15}
\]

As in the previous section let \( \hat{m} = \frac{2w^1}{r^1} \) be the non differential point of the utility function. It is clear that if we fix the exogenous debt constraint \( m = \hat{m} \), then agent 1 problem is convex and we can use standard techniques to prove equilibrium existence. So the real problem is when \( m > \hat{m} \).

Let \( Q_0 \) be the set of equilibrium prices when agent 1 is restricted to \( y \leq \hat{m} \). For all \( q \in Q_0 \) there is a unique portfolio \( y(q) \) that solves debtor’s problem.

---

\(^4\)Actually we can choose the endowments such that agent 1 is the "natural" debtor of the economy, but as we will see later it is important to consider the cases in which this "natural" debtor actually wants to save.

\(^5\)For most of the following results we only need \( C^2 \), but it is useful to restrict to \( C^3 \) to study some properties of the inverse demand function and the \( \hat{m}(\cdot) \) function.

\(^6\)For the natural lender (agent 2), our results also hold when she is risk neutral. Otherwise, when she is not risk neutral nor satisfy Inada conditions, we need to impose additional conditions to guarantee that the inverse demand function is well defined.

\(^7\)It is possible to show that this set is finite.
Now, in $y \geq \hat{m}$ debtor’s utility function is strictly increasing and it is clear that $u^1(y(q)) \geq u^1(\hat{m})$ for all $q \in Q_0$. By continuity there is a $\hat{m}(q) \geq \hat{m}$ such that the debtor is indifferent between this default portfolio and the non default portfolio $y(q)$. Now let $\hat{m} = \max \{\hat{m}(q) : q \in Q_0\}$, then if $m \leq \hat{m}$ there is equilibrium without default.

**Definition.** For all portfolios $m \geq 0$ let $q(m)$ be the inverse demand of agent 2. That is, portfolio $y = m$ is optimal for agent 2 if asset price is $q(m)$.

**Lemma 3.1** The inverse demand $q : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined. It is strictly decreasing and $q(m) = \frac{C}{m}$ for all $m \geq \hat{m}$, where $C$ depends on agents endowments and the seizure rate.

**Proof** Given a portfolio $y = m$, creditor is willing to accept it if it satisfies:

$$-q \frac{du^2}{dx}(w_0^2 - qm) + \sum_{s=1}^{S} \pi_s r_s \kappa_s \frac{du^2}{dx}(w_s^2 + \kappa_s m) = 0,$$

where $\kappa_s = \min \left\{1, \frac{\hat{m}}{m} \right\}$. If $m < \hat{m}$, then we need to find a price $q$ such that:

$$f(q, m) = -q \frac{du^2}{dx}(w_0^2 - qm) + \sum_{s=1}^{S} \pi_s r_s \frac{du^2}{dx}(w_s^2 + r_s m) = 0$$

Note that $f(0, m) > 0$ and $f(q, m) \to -\infty$ when $q \to \frac{w_0^2}{y}$. So there is a $q$ that solves $f(q, m) = 0$, also note that $\frac{\partial f(q, m)}{\partial q} < 0$, so there is a unique $q(m)$ with $f(q(m), m) = 0$. By the implicit function theorem $q(\cdot)$ is $C^1$ and

$$\frac{dq}{dm}(m) = -\frac{\frac{\partial f(q(m), m)}{\partial m}(q(m), m)}{\frac{\partial f(q(m), m)}{\partial q}(q(m), m)} < 0$$

If $m > \hat{m}$, $q(m)$ solves:

$$-q \frac{du^2}{dx}(w_0^2 - qm) + \sum_{s=1}^{S} \pi_s r_s \hat{m} \frac{du^2}{m dx}(w_s^2 + r_s \hat{m}) = 0$$

Define the function $g : [0, w_0^2] \to \mathbb{R}_+$ by $g(z) = \frac{z}{dx}(w_0^2 - z)$. $g$ is strictly increasing, $g(0) = 0$ and $\lim_{z \to w_0^2} g(z) = +\infty$, so $g([0, w_0^2]) = \mathbb{R}_+$. Then, $g(m)m = g^{-1} \left( \sum_{s=1}^{S} \pi_s r_s \hat{m} \frac{du^2}{dx}(w_s^2 + r_s \hat{m}) \right)$ is well defined and $q(m) = \frac{C}{m}$.
where \( C = g^{-1}\left( \sum_{s=1}^{S} \pi_s r_s \bar{m} \frac{dw_s^2}{dx}(w_s^2 + r_s \bar{m}) \right) \) depends on agents endowments and the seizure rate\(^8\).

Now we can define an auxiliary function \( \bar{m}(\cdot) \) as in the previous section:

**Definition.** Let \( \bar{m} : \mathbb{R}_+ \to \mathbb{R}_+ \) be defined as follows: Given a portfolio \( y = m \), \( q(m) \) is the price that supports \( m \) as the optimal choice for agent 2. Let \( y(q(m)) \) be the optimal portfolio for agent 1 when price is \( q(m) \) and she is restricted to \( y \leq \hat{m} \). \( \bar{m}(m) \geq \hat{m} \) is the default portfolio such that agent 1 is indifferent between portfolio \( y(q(m)) \) and \( \bar{m}(m) \).

The following theorem generalizes the result given in the previous section for a general utility function:

**Theorem 3.2** For a given exogenous debt constraint \( m \). The economy has an equilibrium if, and only if, \( m \notin (\bar{m}, \bar{m}(m)) \).

The next proposition characterizes the function \( \bar{m}(\cdot) \) and defines the cases where equilibrium might exist.

**Proposition 3.3** The function \( \bar{m}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) has the following properties:

a) \( \bar{m}(\cdot) \) is strictly increasing for \( m \) large enough.

b) \( \lim_{m \to +\infty} \frac{\bar{m}(m)}{m} = +\infty \)

c) Generically, the function \( f(m) = \bar{m}(m) - m \) has a finite number of zeros.

d) Let \( f(m, w_0^1) = \bar{m}(m, w_0^1) - m \), if \( f \) is a convex function and if debtor initial wealth \( w_0^1 \) is large enough then \( \bar{m}(m, w_0^1) - m > 0 \) for all \( m \).

**Proof**

a) First note that \( \bar{m}(\cdot) \) is well defined because \( y(m) \) is optimal in \( y \leq \bar{m} \) and agent’s objective function is strictly increasing in \( y \geq \bar{m} \). \( \bar{m}(m) \) solves:

\[
\begin{align*}
\pi_s u_s^1(w_s^1 + q(m)y(m)) + \sum_{s=1}^{S} \pi_s u_s^1(w_s^1 - r_s y(m)) &= \pi_s u_s^1((1-\gamma)w_s^1).
\end{align*}
\]

By the implicit function theorem \( \bar{m}(\cdot) \) is a \( C^2 \) function and its derivative satisfies:

\[
q(m) \frac{du_1}{dx}(w_0^1 + q(m)\bar{m}) \frac{d\bar{m}}{dm}(m) = \frac{dq}{dm}(m) \left( y(m) \frac{du_1}{dx}(w_0^1 + q(m)y(m)) - \bar{m} \frac{du_1}{dx}(w_0^1 + q(m)y(m)) \right),
\]

we know that \( q(m) \) is strictly decreasing, so we would like to show that for large values of \( m \) the term inside the parenthesis is negative.

\(^8\)It is worth noticing that \( C \) doesn’t depend on \( w_0^1 \)
Firs note that if the asset price is low enough:

\[ q \leq \sum_{s=1}^{S} \pi_s r_s \frac{du^1}{dx}(w^1_s) \]

then the optimal portfolio is non positive \( y(q) \leq 0 \). There is some \( \tilde{m} \) such that \( q(m) \) satisfies this property for all \( m \geq \tilde{m} \). For all such \( m \), the term inside the parenthesis would be negative, proving that for all \( m \) large enough \( \tilde{m}(\cdot) \) is strictly increasing.\(^9\)

b) First note that by the first order condition \( y(m) \to -\infty \) when \( m \to +\infty \), then by the definition of \( \tilde{m}(\cdot) \) in (16) and by the fact that \( u^1(x) \) goes to +\( \infty \) when \( x \to +\infty \), we conclude that \( \frac{\tilde{m}(m)}{m} \to +\infty \).

c) We can write \( f(m, w^1_0) = \tilde{m}(m, w^1_0) - m \). Let \((m, w^1_0)\) be such that \( f(m, w^1_0) = 0 \). Using the definition of \( \tilde{m}(m) \) and \( y(m) \) and the fact that \( q(m) \) is independent of \( w^1_0 \) we have that \( \frac{\partial f}{\partial w^1_0}(m, w^1_0) > 0 \). So \( f \nless 0 \)\(^{10}\), then by the transversality theorem \( f_{w^1_0} \nless 0 \) generically in the space of endowments, this implies that \( f_{w^1_0}^{-1}(0) \) is empty or discrete. By the previous item we known that \( f_{w^1_0}^{-1}(0) \) is contained in a compact set and we conclude that this set must be empty or finite.

d) Consider the following optimization problem:

\[
\begin{align*}
\text{maximize} & \quad w \\
\text{subject to} & \quad f(m, w) \leq 0
\end{align*}
\]

The function \( f \) satisfies all hypothesis of lemma (A.1) so there is \( w^* \) that solves the optimization problem. Then for all \( w^1_0 > w^* \) it holds that \( f(m, w^1_0) > 0 \) for all \( m \).

**Corollary 3.4** If the initial wealth of the debtor \( w^1_0 \) is large enough then the only possible equilibria are the ones without default.

\(^9\)Actually our estimate is quite conservative, we could find some \( m^+ \) such that \( y(m^+) > 0 \) and \( \tilde{m}(\cdot) \) is strictly increasing for \( m \geq m^+ \). What’s more, if utility function is CES \( u(x) = \frac{x^{1-\sigma}}{1-\sigma} \), then \( \tilde{m}(\cdot) \) is strictly increasing for all \( m \): the term inside the parenthesis would be:

\[
(1-\sigma)\frac{yu^1(w^1_0 + q\tilde{m})}{w^1_0 + q\tilde{m}} - (1-\sigma)\frac{\tilde{m}u^1(w^1_0 + q\tilde{m})}{w^1_0 + q\tilde{m}}
\]

and for \( y \geq 0 \):

\[
(1-\sigma)u^1(w^1_0 + q\tilde{m}) \left( \frac{y}{w^1_0 + q\tilde{m}} - \frac{\tilde{m}}{w^1_0 + q\tilde{m}} \right) < 0
\]

\(^{10}\)\( f \nless 0 \) means that \( Df(m, w^1_0) \) has full rank whenever \( f(m, w^1_0) = 0 \)
Corollary 3.5 There is some $m^\ast$ such that if the economy has an equilibrium with default then $m \leq m^\ast$

The roots of $f$ are important since they define the intervals where equilibrium exists with default. If $f$ has $n$ roots $a_1, a_2, \ldots, a_n$, then for all intervals of the form $[a_{2j-1}, a_{2j}]$ there is equilibrium with default, while the others represents transition intervals without equilibrium.

4 Partial Default and Several Creditors

In this section we make some extensions to the model studied in the previous sections. First, we include the possibility of partial default, also we include many creditors and we take more general utility functions.

Debtor endowment in $t = 1$ depends on the state of nature, i.e. $w \in \mathbb{R}^{S+1}_+$, her preference take the expected utility form with index utility $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $u \in C^1$, strictly increasing, strictly concave, $\lim_{x \rightarrow 0} u'(x) = +\infty$ and $\lim_{x \rightarrow +\infty} u'(x) = 0$.

She chooses a consumption bundle $x \in \mathbb{R}^{S+1}_+$ and a portfolio $y \geq 0$ in order to optimize her expected utility. Again, in case of bankruptcy, a fraction $\gamma$ of her wealth is seized to partially repay the creditors.

Normalizing the price of consumption good to $p = 1$, given the asset price $q$ and the exogenous limit of debt $m_{ex}$, debtor’s optimization problem is:

\[
\begin{align*}
\text{maximize} & \quad u(x_0) + \sum_{s=1}^{S} \pi_s u(x_s) \\
\text{subject to} & \quad x_0 \leq w_0 + qy \\
& \quad x_s \leq \max\{w_s - r_s y; (1 - \gamma)w_s\} \quad \text{for all } s \\
& \quad y \leq m_{ex}
\end{align*}
\]

For each $s = 1 \ldots S$, let $\hat{m}_s = \frac{w_0}{r_s}$. We assume without loss of generality $\hat{m}_1 \leq \hat{m}_2 \leq \cdots \leq \hat{m}_S$.

The next lemma shows that even with the non continuity of the marginal utility, the optimal value function is differentiable at the optimal portfolio, this result has the flavor of the envelope theorem. In [5] the authors shows a general

\[11\text{Although we are assuming exogenously which one is the debtor agent, we can do it endogenously, choosing endowments such that the debtor agent is “poor” in the initial period and “wealthy” in the second one. Actually, endowments should satisfy:} \]
\[\sum_{s=1}^{S} \pi_s r_s (u'(w_0) - u'(w_s)) > 0 \]
\[\text{a sufficient condition for this is } w_0 < w_s \text{ for all } s.\]
form of the envelope theorem when the objective function is constructed out of endogenous functions using standard mathematical operations, their results can be directly applied to our context, but we prefer to give a direct and simple argument.

**Lemma 4.1** Given the asset price \( q \), if the short selling constraint is not binding the objective function is differentiable at the optimal portfolio.

**Proof** Consider the set of points \( \{ \hat{m}_s : s = 1, \ldots, S \text{ and } r_s \pi_s > 0 \} \). We can rewrite the optimization problem such that the objective function depends only on the portfolio \( y \geq 0 \).

\[
U(y) = u(w_0 + qy) + \sum_{s=1}^{S} \pi_s u(\max \{ w_s - r_s y; (1 - \gamma) w_s \})
\]

Note that this objective function is differentiable in all points except those in the set that we’ve just defined. So our task is to show that none of this points is optimal.

Suppose, by contradiction, that \( \hat{m}_s \) is an optimal portfolio. In particular it would be optimal in the intervals \([ \hat{m}_{s-1}, \hat{m}_s] \) and \([ \hat{m}_s, \hat{m}_{s+1}] \) (with \( \hat{m}_0 = 0 \)).

Then, by the first order conditions:

\[
q u'(w_0 + q\hat{m}_s) - \sum_{s'=s}^{S} \pi_{s'} r_{s'} u'(w_{s'} - r_{s'} \hat{m}_s) \geq 0
\]

\[
q u'(w_0 + q\hat{m}_s) - \sum_{s'=s+1}^{S} \pi_{s'} r_{s'} u'(w_{s'} - r_{s'} \hat{m}_s) \leq 0
\]

(19)

so,

\[
0 < \pi_s r_s u'(w_s - r_s \hat{m}_s) \leq qu'(w_0 + q\hat{m}_s) - \sum_{s'=s+1}^{S} \pi_{s'} r_{s'} u'(w_{s'} - r_{s'} \hat{m}_s) \leq 0
\]

a contradiction. ■

Let \( I = \{1, \ldots, I\} \) be the set of creditors, each one chooses a consumption bundle \( x^i \in \mathbb{R}^{S+1}_+ \) and a portfolio \( y^i \geq 0 \) to optimize their expected utility, they take into account that the debtor might be bankrupt in some states of nature and in that case the garnished resources are distributed proportionally among creditors. So, given asset price \( q \) and the mean reimbursement rate \( \kappa \in [0, 1]^{S} \) each creditor solves:
maximize \( u^i(x^i_0) + \sum_{s=1}^S \pi_s u^i(x^i_s) \)
subject to \( x^i_0 \leq w^i_0 - qy^i \)
\( x^i_s \leq w^i_s + \kappa_s r_s y^i \) for all \( s \)

An equilibrium is a vector of consumption \((x^*, (x^*_i)_{i \in I})\), portfolios \((y^*, (y^*_i)_{i \in I})\), asset price \(q^*\) and mean reimbursement rate \(\kappa^* \in [0, 1]^S\) such that:

- Given price \(q^*\), \(y^*\) solves the debtor problem.
- Given price \(q^*\) and \(\kappa^*\), \(y^*_i\) solves creditor problem for all \(i \in I\).
- Market clearing for consumption and assets:
  \[ \sum_{i \in I} x^*_i + x^* = \sum_{i \in I} w^i + w \]
  \[ \sum_{i \in I} y^*_i = y^* \]
- Perfect foresight for all \(s = 1, \ldots, S\):
  \[ \kappa^*_s r_s y^* = \min \{ r_s y^*; \gamma w_s \} \]

### 4.1 Equilibrium without default and endogenous debt constraints

As we mention in the introduction, one way to find equilibrium in models with default is to introduce some "individual rational constraints" like in [4], find and endogenous debt constraint and at the same time avoid default in equilibrium even when this is a feasible choice.

Suppose that we fix the exogenous debt constraint \(m_{ex}\) to \(\hat{m}_1\), debtor agent is not allowed to fill for bankruptcy and her budget constraint is a convex set, so using standard arguments we have the following:

**Proposition 4.2** If the exogenous debt limit is \(m_{ex} = \hat{m}_1\) then there is an equilibrium without bankruptcy.

Now, let \(Q_1\) be the set off all equilibrium prices when \(m_{ex} = \hat{m}_1\). For all \(q \in Q_1\) there is only one portfolio \(y_q\) that solves debtor’s problem. Define the point to set function \(M_1 : Q_1 \mapsto \mathbb{R}_+\) as

\[ M_1(q) = \left\{ m \in \mathbb{R}_+ : m \geq \hat{m}_1, \text{ and } y_q \in \arg\max_{0 \leq y \leq m} U(y) \right\} \]

with \(U(y)\) as in the proof of lemma (4.1).

---

\(^{12}\)In general there might be more than one equilibrium, for instance there is a trivial equilibrium when creditors are too pessimistic, and there is also at least one nontrivial equilibrium, for now on we focus only in non trivial equilibrium.
Lemma 4.3 $\mathcal{M}_1$ has nonempty, compact and convex values.

Proof Note that $\hat{m}_1 \in \mathcal{M}_1(q)$ for all $q$. To see compactness we will prove that $\mathcal{M}_1(q)$ is closed and bounded.

- $\mathcal{M}_1(q)$ is closed. Take a sequence $\{m_k\}_{k \in \mathbb{N}} \in \mathcal{M}_1(q)$ with $m_k \to m$. Take some $y < m$, there is some $k$ such that $y < m_k$, then $U(y_q) \geq U(y)$ and since $U(y_q) \geq U(m_k)$ for all $k$, then by continuity $U(y_q) \geq U(m)$, this proves that $m \in \mathcal{M}_1(q)$.

- $\mathcal{M}_1(q)$ is bounded. Suppose there is a sequence $\{m_k\}_{k \in \mathbb{N}} \in \mathcal{M}_1(q)$ with $m_k \to +\infty$. Then $y_q$ is optimal in $\mathbb{R}_+$, which contradicts the fact that $U(y)$ is unbounded.

Convexity is trivial since if $m \in \mathcal{M}_1(q)$ then all $m' \in [\hat{m}_1, m]$ are also in $\mathcal{M}_1(q)$.

Now we can define the endogenous debt limit as

$$\overline{m}_1(q) = \max \{m \in \mathcal{M}_1(q)\}$$

It is endogenous because it depends on equilibrium variables. Now we can state the following theorem whose proof follows from the previous arguments.

Theorem 4.4 If $\hat{m}_1 < m_{ex} \leq \overline{m}_1(q)$ for some $q \in Q_1$, then there exists an equilibrium without default and with non convex budget constraint.

This result is important in the sense that we can find an equilibrium even with non convexities in the budget constraint, debtor is allowed to fill for bankruptcy at least in one state of nature, but she still prefers to honor her debt. Note that the last theorem requires that $\overline{m}_1(q) > \hat{m}_1$, this would be true when $y_q$ is an interior optimal in $[0, \hat{m}_1]$ as stated in the following

Lemma 4.5 $\overline{m}_1(q) > \hat{m}_1$ if, and only if, $y_q < \hat{m}_1$.

Proof This is an easy consequence of lemma 4.1, if $\hat{m}_1 < \overline{m}_1(q)$, then by lemma 4.1 $y_q < \hat{m}_1$. Conversely, there is an $\epsilon > 0$ such that $U(y_q) > U(\hat{m}_1 + t)$ for all $0 \leq t \leq \epsilon$, then $\hat{m}_1 + \epsilon \in \mathcal{M}_1(q)$, and $\overline{m}_1(q) \geq \hat{m}_1 + \epsilon > \hat{m}_1$.

4.2 Equilibrium with Default

Although the economy we just described allow filling for bankruptcy, in equilibrium there is not default, now we try to find equilibria with default. In order to do this we need to chose an exogenous debt constraint bigger that the endogenous constraint that we found in the last section, that is $m_{ex} > \overline{m}_1(q)$ for all $q \in Q_1$.

Before, we need an auxiliary definition of equilibrium that we call sub-equilibrium with controlled bankruptcy. This definition is inspired in the simple observation that the non-convexity problem arises because the default states are determined endogenously. If we restrict exogenously the bankruptcy states and forbid debtor to change this, then we can get rid of the non-convexity problem.
Definition. Consider the interval \( I_s = [\hat{m}_{s-1}, \hat{m}_s] \) for all \( s = 1, \ldots, S+1 \) (with \( \hat{m}_0 = 0 \) and \( \hat{m}_{S+1} = m_{ex} \)). An \( s \) sub-equilibrium is an equilibrium in which the portfolio of the debtor is restricted to \( I_s \).

Remark. In \( I_s \) debtor’s optimization problem is well behaved, in the sense that the objective function is concave and the feasible set is convex.

The next proposition is proved by standard arguments and is in the same spirit of proposition 4.2.

Proposition 4.6 For all \( s = 1, \ldots, S+1 \) there is an \( s \) sub-equilibrium.

For all \( s \) let \( Q_s \) be the set of all \( s \) sub-equilibrium prices. Given \( q_s \in Q_s \) let \( y_{q_s} \) be the optimal portfolio in \( I_s \). The main issue is that given this price, it might happen that \( y_{q_s} \) is not optimal for the original problem.

Theorem 4.7 Consider the following point to set function \( \Gamma : S^* \rightrightarrows S^* \) (where \( S^* = \{1, \ldots, S+1\} \)) as:

\[
\Gamma(s) = \bigcup_{q \in Q_s} \left\{ s' \in S^* : \exists z \in I_{s'} \text{ s.t. } z = \arg\max_{0 \leq y \leq m_{ex}} U(y, q) \right\}
\]

Then, \( s \in \Gamma(s) \) if, and only if, there is an equilibrium with debtor’s portfolio in \( I_s \).

Proof \( \Rightarrow \) Let \( q \in Q_s \) and \( z \in I_s \) such that \( z \) solves debtor problem when asset price is \( q \). Consider the \( s \) sub-equilibrium associated with \( q \), then there is an equilibrium portfolio \( y^* \in I_s \), so \( U(y^*, q) \geq U(z, q) \) and since \( y^* \leq m \) then \( U(z, q) \geq U(y^*, q) \), so \( y^* \) solves debtors problem, then the \( s \) sub equilibrium is actually an equilibrium.

\( \Leftarrow \) Consider the equilibrium with portfolio \( y^* \in I_s \) and asset price \( q \), clearly \( y^* \) is optimal in \( I_s \) at price \( q \), then it is an \( s \) sub-equilibrium and \( q \in Q_s \) and \( s \in \Gamma(s) \).

Corollary 4.8 If \( m_{ex} \leq \bar{m}_1(q) \) for some \( q \in Q_1 \), then \( \Gamma \) has a fix point.

In order to understand the correspondence \( \Gamma \) we will study the behavior of the \( s \) sub-equilibria. It is important to know how much \( I_s \) can be expanded without changing the optimal choice, to do this we define for all \( s = 1, \ldots, S+1 \) the correspondences \( \mathcal{M}_s : Q_s \rightrightarrows \mathbb{R}_+ \) and \( \mathcal{W}_s : Q_s \rightrightarrows \mathbb{R}_+ \) as follows:

\[
\mathcal{M}_s(q) = \left\{ m \in \mathbb{R}_+ : m \geq \hat{m}_s, y_q \in \arg\max_{\hat{m}_{s-1} \leq y \leq m} U(y, q) \right\}
\]

and

\[
\mathcal{W}_s(q) = \left\{ m \in \mathbb{R}_+ : m \leq \hat{m}_{s-1}, y_q \in \arg\max_{m \leq y \leq \hat{m}_s} U(y, q) \right\}
\]
where, for \( q \in Q_s \), \( y_q \in I_s \) is the only optimal choice for debtor agent when he is restricted to \( I_s \) and asset price is \( q \).

As in lemma 4.3 we can prove the following

**Lemma 4.9** For all \( s = 1, \ldots, S + 1 \), \( M_s \) and \( W_s \) have compact, convex and non empty values. And

\[
\overline{m}_s(q) = \max_{m \in M_s(q)} m \\
\underline{m}_s(q) = \min_{m \in W_s(q)} m
\]

are well defined.

**Remark.** \([\underline{m}_s(q), \overline{m}_s(q)]\) is the biggest interval such that \( y_q \) is an optimal choice.

**Remark.** By lemma 4.1 it is clear that \( I_s \subseteq [\underline{m}_s(q), \overline{m}_s(q)] \) with proper inclusion for \( s = 2, \ldots, S + 1 \).

**Remark.** For \( s = 1 \), \( W_1(q) = \{0\} \) for all \( q \in Q_1 \) and for \( s = S + 1 \), \( M_{S+1}(q) = \{m_{ex}\} \) for all \( q \in Q_{S+1} \).

The next theorem characterizes all the situations in which equilibrium might exist, note that if it holds for some \( s \neq 1 \) then there is an equilibrium with default and non convex budget set.

**Theorem 4.10** \( s \) is a fix point of \( \Gamma \) if, and only if, there is some \( q \in Q_s \) such that \( m_s(q) = 0 \) and \( m_{ex} \leq \overline{m}_s(q) \).

According to the last theorem, if \( m_s(q) > 0 \) for all \( q \in Q_s \) and all \( s = 2, \ldots, S + 1 \) then the only possible equilibrium is one that avoids default. In some cases it is known like in [6] that default might be efficient, to recover this efficiency in our context, in the next section we introduce a regulation in consumption that allows equilibrium with default even if the mentioned condition holds.

Next we show that in order to have an equilibrium, the exogenous debt constraint \( m_{ex} \) can’t take large values. The intuition is simple, as \( m_{ex} \) gets larger, the asset price that supports this portfolio as a sub-equilibrium gets smaller because creditors are willing to buy this asset that always default only if they are compensated with a large interest rate (or equivalently a low asset price), but if asset prices are too low, then debtor is not willing to hold a large portfolio and she has incentives to deviates from the full default strategy.

Before we announce our theorem, we need the following

**Lemma 4.11** If the initial wealth of debtor is large enough, then there is some \( m_{ex}^* \) such that \( 0 \notin W_{S+1}(q) \) for all \( q \in Q_{S+1} \) and all \( m_{ex} \geq m_{ex}^* \).
Proof Take an $m \geq \hat{m}_S$, then, if debtor offers this portfolio she defaults in all states, so $\kappa_s = \gamma w_s / (r_s m)$ for all $s = 1, \ldots, S$.

Given $(q, m)$, portfolio $y_i \geq 0$ is optimal for creditor $i$ if, and only if,

$$-qu'_i(w_{i0} - qy_i) + \sum_{s=1}^S \pi_s \kappa_s r_s u'_i(w_{is} + \kappa_s r_s y_i) \leq 0$$

with equality if $y_i > 0$.

Let $y_i(q, m)$ be creditor $i$’s demand function, note that it is continuous in both arguments. Now, given debtor’s offer $y = m \geq \hat{m}_S$, creditors are willing to buy this asset if $q$ solves

$$h(q) = \sum_{i \in I} y_i(q, m) - m = 0.$$ 

By continuity arguments it follows that there is a $q(m)$ that solves $h(q) = 0$.

Take a sequence $m_n \rightarrow \infty$. Note that $y_i(q(m_n), m_n) / m_n$ is bounded and for at least one $i_0$, $y_{i_0}(q(m_n), m_n) \rightarrow 0$ holds. Inada condition guarantees that

$$w_{i_0} - q(m_n) y_{i_0}(q(m_n), m_n) > 0$$

for all $i \in I$, in particular for $i_0$, then $q(m_n) \rightarrow 0$.

Next we notice that in each interval $[\hat{m}_s, \hat{m}_{s+1}]$, the lower bound is optimal if asset price is low enough, then if we take a credit limit $m$ large enough, the lower bound is optimal in each such interval, but this means that portfolio $y = 0$ is optimal in $[0, \hat{m}_S]$.

Summing up, if we take a credit limit $m$ large enough, we need to compare the autarky utility with the full default utility. Then, $0 \not\in W_{S+1}(q)$ if

$$0 < \Delta = u(w_0) + \sum_s \pi_s u(w_s) - \left( u(w_0 + q(m)m) + \sum_s \pi_s (u((1 - \gamma)w_s)) \right)$$

By concavity:

$$\Delta \geq q(m)m \left[ \frac{1}{q(m)m} \sum_s \pi_s (u(w_s) - u((1 - \gamma)w_s)) - u'(w_0) \right],$$

which is positive for $w_0$ large enough.

Now we can state the following theorem that shows a necessary condition for equilibrium existence.

**Theorem 4.12 (Necessary condition)** Suppose that the debtor initial wealth is large enough, if $\Gamma$ has a fix point, then the exogenous debt constraint $m_{ex}$ can’t take large values.
Proof Let \(m^*_e\) be as in lemma 4.11. Define the set of indexes \(\hat{S}\) as:
\[
\hat{S} = \{s \in \{1, \ldots, S\} : 0 \in W_s(q) \text{ for some } q \in Q_s\}
\]
Note that this definition doesn’t depend on the choice of \(m_e\). Define
\[
m_1 = \max_{s \in \hat{S}, q \in Q_s} \tilde{m}_s(q)
\]
and \(m^* = \max \{m_1, m^*_e\}\). We claim that if \(m_e > m^*\) then there are not equilibria. First note that no \(s\) sub-equilibrium with \(s \notin \hat{S}\) can be implemented as an equilibrium.

For \(s \in \hat{S}\), since \(m_e > \tilde{m}_s(q)\) then the portfolio of sub-equilibrium \(y_q\) is no longer optimal in \([0, m_e]\). So the only possible equilibrium is one with full default, but since \(m_e > m^*_e\), by lemma 4.11 we have that \(0 \notin W_{S+1}(q)\), then the full default portfolio \(y = m_e\) is not optimal. This shows that there are no equilibria if the exogenous debt constraint is large enough.

4.3 Example with a risk neutral creditor

Suppose there is one debtor and one risk neutral creditor, then creditor optimization problem is:
\[
\max_{y \geq 0} \ w_0^c + \sum_{s=1}^{S} \pi_s w_s^c + \left(\sum_{s=1}^{S} \pi_s \kappa_s r_s - q\right) y
\]
so, asset price must satisfy
\[
q \geq \sum_{s=1}^{S} \pi_s \kappa_s r_s \quad (21)
\]
with equality if \(y > 0\). Note that when there is no default, asset pricing follows the same rule as in the classic asset pricing theory, but when defaults do occur, then asset returns are given endogenously and asset price might be lower than the expected value of the promise.

To find the \(s\) sub-equilibrium we must solve the debtor’s optimization problem:
\[
\max_{m_{s-1} \leq y \leq m_s} u(w_0 + qy) + \sum_{1 \leq s' \leq s-1} \pi_{s'} u((1 - \gamma)w_{s'}) + \sum_{s' \geq s} \pi_{s'} u(w_{s'} - r_{s'} y),
\]
so, we must find \((y, q)\) that solves:

\[
q u'(w_0 + q y) - \sum_{s' \geq s} \pi_{s'} r_{s'} u'(w_{s'} - r_{s'} y) - \lambda + \mu = 0,
\]

\[
q = \frac{\gamma}{y} \sum_{1 \leq s' \leq s-1} \pi_{s'} w_{s'} + \sum_{s' \geq s} \pi_{s'} r_{s'},
\]

\[
\lambda (\hat{m}_s - y) = 0, \mu (y - \hat{m}_{s-1}) = 0,
\]

\[
\hat{m}_s - y \geq 0, \lambda \geq 0,
\]

\[
y - \hat{m}_{s-1} \geq 0, \mu \geq 0.
\]

Consider the extreme cases of no default and full default, in the former case the asset price, as we mentioned before, is just the expected value of the promise and the seizure rate \((\gamma)\) only affect the upper portfolio constraint, it is easy to see\(^\text{13}\) that when \(\gamma\) is large enough this constraint is not biding and the optimal values are independent of the seizure rate. On the other hand, in the full default sub-equilibrium, portfolio value is given by:

\[
q y = \gamma \sum_{s=1}^{S} \pi_s w_s,
\]

and optimal utility is a concave function of the seizure rate:

\[
u_{\gamma}\left(\sum_{s=1}^{S} \pi_s (w_0 + \gamma w_s)\right) + \sum_{s=1}^{S} \pi_s u((1 - \gamma) w_s),
\]

in the case of log utility and provided that \(\bar{w} = \sum_{s=1}^{S} \pi_s w_s > w_0\), the last expression attains its maximum at \(\gamma^* = \frac{2\bar{w}}{2\bar{w} - w_0} \leq \frac{1}{2}\) which means that a regulator who wants to implement a sub-equilibrium with full default should choose a seizure rate below \(1/2\), and if the bankruptcy law punishes the debtor too hard (high values of seizure rate), then the optimal utility in the full default sub-equilibrium will be too low and the debtor would probably do better in a sub-equilibrium with partial default or full commitment. Figure 5 shows the optimal utility in the full default sub-equilibrium as a function of the seizure rate in the case of 3 states of nature and endowment \(w = (0.1, 0.8, 0.9, 1)\), the optimal is below \(1/2\) and the graph is asymmetric around this value, which means that the debtor values differently increases and decreases of the seizure rate.

### 4.3.1 Optimal default with low seizure rate

In the extreme case of \(\gamma = 0\) debtor will always choses to default in all states of nature, she has no punishment for default and her utility function is strictly

\[^{13}\text{Note that:}
q u'(w_0 + q \frac{w_1}{r_1}) - \pi_1 r_1 u'(1 - \gamma) w_1 - \sum_{s=2}^{S} \pi_s r_s u'(w_s - \gamma w_1 \frac{r_s}{r_1}) < 0
\]

for all \(\gamma\) close enough to one, then the upper constraint can’t be biding.
increasing in portfolio. Figure 6 shows the utility function for different values of the seizure rate, all curves are between the two extremes $\gamma = 0$ and $\gamma = 1$ and it is seen that for low values of the seizure rate, utility function is strictly increasing.

Table 1 shows optimal utility, portfolios and prices for all sub-equilibria, the last column shows the interval $[\underline{m}_s, \overline{m}_s]$ defined in the previous section, this interval is useful to determine whether the $s$ sub-equilibrium is an equilibrium. As we mentioned, for low values of the seizure rate, the utility function is strictly increasing, so the upper constraint will be binding and the upper bound can’t be expanded without changing the equilibrium as can be seen in the last columns for values of $\gamma$ below 0.45.

Note that for a low, but fix seizure rate, the non-default sub-equilibrium is implementable as an equilibrium if we let the exogenous debt constraint be small enough, but this equilibrium can be Pareto improved if we let the debtor default in one or more states, actually the best equilibrium (in a Pareto sense) is the one with default in states one and two.\(^{14}\) In all sub-equilibria, the upper constraint is bidding and the prices are the same in each sub-equilibrium.

Summing up, if we start from a low seizure rate, a regulator can Pareto improve equilibrium if he increases the seizure rate and allows default and at the same time he can increase the amount of investment in portfolio without affecting prices.

When we increase the seizure rate, the non-default equilibrium corresponds to an interior portfolio and the optimal utility is independent of the seizure rate. We are able to extend the interval in which this portfolio is optimal (see the last column of table 1) and this does depends on the seizure rate, as it gets higher, the interval also gets bigger, meaning that as the default penalty gets

\(^{14}\) The same utility can be achieved in a full default equilibrium with a not too big exogenous debt constraint, as we argue before, in this case the optimal utility remains constant as we change the exogenous debt constraint.
Figure 6: Utility function for different values of the seizure rate

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Sub-equilibrium</th>
<th>Optimal Utility</th>
<th>$y$</th>
<th>$q$</th>
<th>$[m_1, m_2]$</th>
</tr>
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<tbody>
<tr>
<td>0.15</td>
<td>1</td>
<td>$-1.7681$</td>
<td>0.12</td>
<td>1</td>
<td>$[0, 0.12]$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$-1.7358$</td>
<td>0.135</td>
<td>0.9630</td>
<td>$[0, 0.1350]$</td>
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<tr>
<td></td>
<td>3</td>
<td>$-1.7202$</td>
<td>0.15</td>
<td>0.9</td>
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</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>$-1.5675$</td>
<td>0.2</td>
<td>1</td>
<td>$[0, 0.2]$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$-1.5362$</td>
<td>0.225</td>
<td>0.9630</td>
<td>$[0, 0.225]$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-1.5211$</td>
<td>0.25</td>
<td>0.9</td>
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</tr>
<tr>
<td>0.35</td>
<td>1</td>
<td>$-1.4544$</td>
<td>0.28</td>
<td>1</td>
<td>$[0, 0.28]$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$-1.4308$</td>
<td>0.315</td>
<td>0.9630</td>
<td>$[0, 0.315]$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-1.4198$</td>
<td>0.35</td>
<td>0.9</td>
<td>$[0, 0.35]$</td>
</tr>
<tr>
<td>0.45</td>
<td>1</td>
<td>$-1.4043$</td>
<td>0.36</td>
<td>1</td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td>$-1.3944$</td>
<td>0.405</td>
<td>0.9630</td>
<td>$[0, 0.405]$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-1.3905$</td>
<td>0.45</td>
<td>0.9</td>
<td>$[0, 0.45]$</td>
</tr>
<tr>
<td>0.55</td>
<td>1</td>
<td>$-1.3997$</td>
<td>0.3933</td>
<td>1</td>
<td>$[0, 0.4593]$</td>
</tr>
<tr>
<td></td>
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<td>$-1.4272$</td>
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<td>0.9630</td>
<td>$[0, 0.495]$</td>
</tr>
<tr>
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<td>0.9</td>
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</tr>
<tr>
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<td>1</td>
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<td>0.3933</td>
<td>1</td>
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<tr>
<td></td>
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<td>0.5263</td>
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</tr>
<tr>
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<td>0.65</td>
<td>0.9</td>
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</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>$-1.3997$</td>
<td>0.3933</td>
<td>1</td>
<td>$[0, 1.0009]$</td>
</tr>
<tr>
<td></td>
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<td>$-1.5999$</td>
<td>0.6</td>
<td>1</td>
<td>$[0.6, 0.8011]$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-1.7281$</td>
<td>0.7229</td>
<td>0.9212</td>
<td>$[0.6699, 0.7519]$</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>$-1.3997$</td>
<td>0.3933</td>
<td>1</td>
<td>$[0, 1.7348]$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$-1.8397$</td>
<td>0.68</td>
<td>1</td>
<td>$[0.68, 1.0816]$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$-2.0353$</td>
<td>0.765</td>
<td>0.9630</td>
<td>$[0.765, 0.9053]$</td>
</tr>
<tr>
<td>0.95</td>
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<td>0.3933</td>
<td>1</td>
<td>$[0, 5.4043]$</td>
</tr>
<tr>
<td></td>
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<td>1</td>
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<tr>
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<td>3</td>
<td>$-2.8301$</td>
<td>0.855</td>
<td>0.9630</td>
<td>$[0.855, 1.2635]$</td>
</tr>
</tbody>
</table>

Table 1: Optimal utility, portfolio and price for all sub-equilibria and for different values of the seizure rate.
stronger, the debtor agent prefers the non default equilibrium for a large range of portfolios.

It is also worth noticing that for large values of $\gamma$, all sub-equilibria that involve default in some state of nature, are not implementable as equilibria.

5 Pareto Efficiency of Bankruptcy Equilibrium

In the previous section we showed that the bankruptcy equilibrium might Pareto dominate the non default equilibrium. Now we explore (as in [2]) the possibility that the bankruptcy equilibrium is Pareto efficient while the non default equilibrium is not.

Consider an economy with two agents $i = 1, 2$ such that agent 1 is the natural debtor and her preferences are described by a continuous, strictly concave and strongly increasing utility function such that $\lim_{x \to 0} u_1'(x) = +\infty$, while agent 2 is risk neutral with utility $u_2(x) = x$. In $t=1$ there are $s = 1, \cdots, S$ states of nature. Agents endowments are:

$$w^i = (w^i_0, w^i_1, \cdots, w^i_S) > 0$$

such that $w^i_s = w$ for all $s = 1, \cdots, S$, that is, agent 1’s endowment is deterministic. We also assume that $0 < w^i_0 < 2w < 2w^i_0$. There is one nominal asset with payoffs $r_1 > r_2 > \cdots > r_S$, then $\hat{m}_1 = \frac{w^i_1}{r_1} < \hat{m}_2 < \cdots < \hat{m}_S$.

We can characterize the Pareto optimal allocation as in [9]. Take some $\lambda \in [0, 1]$ and consider the following problem:

$$\max_{0 \leq x^i_s \leq w_s \text{ for all } s = 0, 1, \cdots, S} \lambda \left( u_1(x^i_0) + \sum_{s=1}^{S} \pi_s u_1(x^i_s) \right)$$

$$+ (1 - \lambda) \left( u_2(w_0 - x^i_0) + \sum_{s=1}^{S} \pi_s u_2(w_s - x^i_s) \right),$$

where $w_s = w^i_1 + w^i_2$ for all $s = 0, 1, \cdots, S$.

The necessary and sufficient condition of optimality are:

$$\lambda u_1'(x^i_s) \geq (1 - \lambda)u_2'(w_s - x^i_s) \text{ with equality if } x^i_s < w_s \text{ for all } s = 0, 1, \cdots, S$$

Since agent 2 is risk neutral, we conclude that an allocation $\{(x^i_s, x^2_s)\}_{s=0, 1, \cdots, S}$ is Pareto efficient if and only if there is a $\lambda \in [0, 1]$ such that for all $s = 0, 1, \cdots, S$:

$$\lambda u_1'(x^i_s) \geq (1 - \lambda) \text{ with equality if } x^i_s < w_s$$
\[ x_2^s = w_s - x_1^s, \]
in particular for all states such that \( x_1^s < w_s, \ u'_1(x_1^s) \) should be the same, and since utility is strictly concave, consumption must be the same in all such states.

Now let’s see what happens in the economy without default. The price should be \( q = \sum_{s=1}^{S} \pi_s r_s \) and debtor agent solves:

\[
\begin{align*}
\text{maximize } & \quad u_1(w_0^1 + qy) + \sum_{s=1}^{S} \pi_s u_1(w - r_s y) \\
\text{since } & \quad w_0^1 < w \text{ we have that } qu'_1(w_0^1) - \sum_{s=1}^{S} \pi_s r_s u'_1(w) > 0, \text{ so the optimal portfolio is } y^* > 0. \text{ Consumption in } t = 0 \text{ is } w_0^1 + qy^* \text{ and in } t = 1 \text{ in state } s \text{ consumption is } x_1^s = w - r_s y^* < w < w_s, \text{ so } x_1^1 < x_2^1 < \cdots < x_S^1 \text{ which implies that this equilibrium is not efficient.}
\end{align*}
\]

When we allow for bankruptcy in all states of nature \((m \geq n_s)\), agent 1 chooses the portfolio \( y = m \) and asset price is \( q = \sum_{s=1}^{S} \pi_s \kappa_s r_s \), with \( \kappa_s = \frac{\gamma w}{r_m} \), that is \( qm = \gamma w \).

Consumption is:

\[ x_0^1 = w_0^1 + qm = w_0^1 + \gamma w < w_0 \]
and for \( s = 1, \cdots, S \):

\[ x_1^s = (1 - \gamma)w < w_s. \]

If we take \( \gamma = \frac{1}{2} - \frac{w_1^1}{w} \in (0, \frac{1}{2}) \), then \( x_1^s = (1 - \gamma)w \) for all \( s = 0, 1, \cdots, S \). Finally we can define \( \lambda = (1 + u_1((1 - \gamma)w))^{-1} \) and conclude that \( \{(x_1^s, x_2^s)\}_{s=0,\ldots,S} \) is Pareto efficient.

The intuition of this results is that debtor faces the problem of choosing between increasing consumption in the first date and uncertainty in the second one. Because she is relatively poor in the first period, she would like to sell the asset, but since she is risk averse she would also like to smooth consumption between states of nature. The exogenous payoff of the given asset is random and her wealth is deterministic, so this asset is not a really good option for her. When we allow for bankruptcy, the span of the asset changes and in the case of default in all states the asset returns are the same in all states of nature eliminating uncertainty for consumption in the second date. If in addition we chose a seizure rate that equals consumption in both dates, then we obtain an efficient equilibrium.
6 Final Remarks

In this paper we address an economy with bankruptcy, unlike other approaches that consider a non-atomic set of agents, we don’t assume that there is a continuum of agents of each type and characterize the conditions that guarantee equilibrium existence with two agents: a creditor and a debtor who is allowed to fill for bankruptcy when her wealth is not enough to pay her debts.

We show that, with log utility function, when debtor’s initial wealth is zero there is always an equilibrium: if the exogenous debt constraint is small enough the equilibrium is the same as in an economy without bankruptcy, otherwise there is an equilibrium with bankruptcy in which debtor short sells as much as she can and fills for bankruptcy in the second period.

We define a threshold for the debt constraint such that debtor goes bankruptcy if, and only if, the exogenous debt constraint is beyond this value. When debtor initial wealth is zero, this threshold independent of asset price because best response correspondence is inelastic, so we can verify before solving for equilibrium whether debtor fills for bankruptcy, equilibrium price can be directly computed from creditor’s problem.

We show that an authority that regulates the credit limit might choose a Pareto improving value. We also show through a numerical example that general equilibrium effects on prices might more than offset the utility increasing of debtor agent when credit limit increases.

If we slightly modify the model such that debtor’s initial wealth is positive, equilibrium might not exists depending on the credit limit: when credit limit is small enough there is an equilibrium without bankruptcy, on the other hand, for large values that might be interpreted as a lack of regulation, equilibrium fails to exits because bankruptcy jeopardize asset prices and in some point market shuts down. In the intermediate cases equilibrium might (or not) exit depending on the debtor initial wealth. Interestingly there is a transition set of credit limits values that separates full commitment equilibrium with bankruptcy equilibrium (if there exist), though it is possible to eliminate this interval if the seizure rate is low enough.

In the case of partial default and several creditors we characterize equilibrium as a fix point of a discrete valued correspondence. We define a concept of sub-equilibrium noticing that fixing the default states, the feasibility set is convex, this sub-equilibrium always exists, but it might not be an equilibrium. As in the deterministic case, equilibrium without bankruptcy exists for low values of the exogenous debt constraint and for high values of the exogenous debt constraint equilibrium doesn’t exist. Finally in our example with a risk neutral creditor we show that the equilibrium with default Pareto dominates the non default equilibrium when the seizure rate is low, moreover, this equilibrium
might be Pareto efficient if we chose the seizure rate properly. This results holds because bankruptcy changes the span of the asset returns and the endogenous returns might be more suitable than the exogenous.

Further research must be done to include more general environments such as asymmetric information, multiperiod settings that allows to study market partial exclusion as a default penalty. Also, more debtor agents should be introduced in order to study systemic risk. More general reimbursement rules could also play an important role to rank equilibria. It also may be interesting to endogenize the credit limits perhaps as a function of the seizure rate.

References


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Appendix A

Lemma A.1  Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be a convex differentiable function such that there is a $\hat{x} \in \mathbb{R}_+^n$ with $\nabla f(\hat{x}) \gg 0$. Then the set $L_f(0) = \{ x \in \mathbb{R}_+^n : f(x) \leq 0 \}$ is compact.

Proof  If $L_f(0) = \emptyset$ the result hold trivially, otherwise take some $x \in L_f(0)$. By convexity:

$$0 \geq f(x) \geq f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x} \rangle,$$

then

$$\langle \nabla f(\hat{x}), x \rangle \leq \langle \nabla f(\hat{x}), \hat{x} \rangle - f(\hat{x}),$$

since $\nabla f(\hat{x}) \gg 0$, $\min \{ \partial_i f(\hat{x}) : i = 1, \ldots, n \} = C > 0$, and because $x \geq 0$, we conclude that for all $i = 1, \ldots, n$:

$$0 \leq x_i \leq \frac{\langle \nabla f(\hat{x}), \hat{x} \rangle - f(\hat{x})}{C} < +\infty.$$